Commentaries – Student Work
Inverses Talk Frame

General comment: One important feature to note is that in arguing for or against the inverse function, it can be helpful for students to reference or offer a definition of the inverse function. Students tend to rely on procedures they have learned (such as "switching the x and y") to show that – based on learned rules – each is or is not the proper inverse of f(x). Encouraging students to link back to the definition of inverses can strengthen such arguments (as that is what they are trying to show is or is not true), provide an opportunity to deepen their understanding of inverse functions, and provide the teacher with valuable information about what the student is and is not yet understanding.

Work Sample - Student A

This student overall does a fairly nice job arguing for or against each of Jeana's, John's and Kevin's representations as accurate representations of the inverse function. There are some aspects of the arguments that may need some attention, however. We discuss each in detail:

The student claims **Jeana** is correct because "when you plug in the inverse numbers with the given function, the answers come out to the original x." The question remains: why is it that this fact (that plugging in the $f^{-1}(x)$ numbers and getting the x values) means you have the correct inverse? A definition of inverse that includes the idea that for all values of a function, if a function maps x1→y1, then the inverse function must map y1 → x1. This explains why the work the student did to test the numbers in Jeana's table in this way is relevant for determining that the function is the correct inverse. Note however that Jeana's table is limited to four values. Noticing that the definition of an inverse function requires this to be true for all x values of the function, one might argue Jeana's is *not* the inverse function, although each point in the table would be part of the inverse function.

To refute **John's** claim of the inverse, the student goes through the process of finding the inverse and notes that she got something different. The implied argument is: if I do the procedure to find the inverse correctly, and get something different than what John got, he must be wrong. The student does not explain what process is being used, why, or how they know it will produce the correct inverse. The teacher may choose to ask for further elaboration: What is an inverse, and why does it make sense that your process would produce the inverse? (These are challenging questions and get to the heart of whether students understand what an inverse is.)

To support the claim that **Kevin**'s graph represents the inverse, the student uses a productive (and efficient!) approach of connecting Kevin's graph to Jeana's data points. The argument basically is: Since Kevin's matches Jeana's, and I already showed Jean's correct, Kevin's right too. This is a valid approach. One point not addressed, however, is that John's line has many more points than Jeana's 4-point table. Does matching on 4 points mean that Kevin has the correct inverse? Should we really have said that Jeana's was the right inverse?

In the discussion, it could be brought out that Jeana showed 4 points on the inverse, but did not represent the full table. As Jeana's table cannot be considered to fully represent $f^{-1}(x)$, there then needs to be more to the argument to show that Kevin's is indeed the right inverse.
Shoring up the argument for Kevin's graph can come from looking at the student's response to John's inverse, as in that argument, the student derived the proper inverse function. The student could argue that Kevin's line represents the inverse function y = (1/3)x +(2/3).

Work Samples - Student B
This student starts with **Kevin's Claim** and appeals to finding the inverse function correctly, presumably on the calculator or using Desmos, and having it match Kevin's claim, so Kevin must be correct. The argument is logical. It does not however compel the reader to accept that Kevin's is correct, as the reader doesn't know what the student did on the calculator (and remains in the dark about what an inverse function should actually do, and so cannot judge for himself whether the argument seems solid).

Jeana's claim is argued as true by linking it to Kevin's graph, and showing that all the points from Jeana's table are on Kevin's graphed inverse line. This is a valid approach, and shows that (assuming we think Kevin's claim is correct) all Jeana's points are part of the inverse. The student does not address that Jeana's inverse may be incomplete.

John's claim of the inverse function is treated as correct (although it is not) "because when you plug points into his claim they do come out correct." The student shows 4 x-y pairs in a table. (We don't know where these came from, but 3 overlap with Jeana's table.) It seems the student derived 4 coordinate pairs from the original function and then perhaps tested John's proposed inverse. Finding all 4 worked, the student claimed John's function was the inverse.
There are two problems to note. First, testing 4 points is not a valid way to demonstrate that something is true for *all points.* It's also not clear *why* the student is plugging in these points and why having them "come out correct" means that the function is an inverse. If one has a good sense of what an inverse function is, then it makes sense for a person to plug these points in. However, for a reader who doesn't know much about inverse functions, these activities would not be understandable – how does that help you decide if it's an inverse function?
Second, because of an order of operations error perhaps that seems to have the student subtracting 2 from the x before dividing by 3, the student thinks these points *are* on the inverse function when they are not. Had the student noticed he should always be subtracting 2/3 from some whole number (given his table), which would result in something that is *not* a fraction, he might have caught the error. This approach is viable however, as if the math were correctly executed, it would have led to the conclusion that John's function was *not* the correct inverse function. Note that, although 4 points cannot show something true, 1 point (one counterexample) can show something is false.

Work Sample – Student C

This student, like others, tested **Jeana's** values. The argument is not articulated however. The student expects the teacher, a reader, someone, can understand that plugging in 2 to the "original equation" and getting out 8, means this is an inverse function. There's no information about why that should be the case, or why the student tested all the values.

John's claim for the inverse is refuted by one counterexample, which is a valid approach. The student explains what they did – plugging in (2,0) – and because it did not work, John's can't be the right inverse. Again what's left unexplained is why plugging in (2,0), and having the result be a true statement, would be what's expected if you were looking at the inverse. What does the inverse do? What's the relationship of an inverse function to the original function? Having information about these questions would strength the argument.

The student's argument that **Kevin's inverse graph** is correct also rests on empirical work –testing values. The same gaps are present as with other arguments: Given that we're looking for an inverse, why do we expect this "switched situation" that seems to be central to what the student is trying to show? And how do you know it holds for *all* values, and not just those you tested (as the inverse function in this case has an infinite number of values)?

Work Sample – Student D

For **Jeana's,** the student argues it is a correct representation of the inverse based on putting the x of the table in for the y of the original. The student shows work to test 2 points. Left unstated is why the inverse and original should show this relationship. The student also asserts it is the inverse based on testing 2 of 4 points, and does not address the fact that the table shows only 4 of an infinite number of points.

For **John's** the student shows all work to create the inverse. No explanation of what the student is doing or why is provided. (The student assumes the reader knows what this process is and will agree that it will produce an inverse function.) Interestingly, the student *critiques* John's approach more directly, but pointing out an error with John's mathematics: "he did not divided x by 3."

For **Kevin's,** the student claims Kevin is correct and offers as evidence 2 tested points. He seems to use the original function to generate an ordered pair (e.g., (1, 5)) and then confirm the point created by switching the x and y (e.g (5,1)) is on the inverse graph. This is repeated with (2,0). Our two usual questions are again relevant here: Why would this process help you know whether this is an inverse? (Why does it make sense to do this math work? What can it tell you?) And how does testing 2 cases show it for all cases?

Overall comments and a point on strengthening arguments.

The idea of an inverse function, at its core, is fairly straightforward: an inverse function "undoes" the original function. It's a function that takes an output value to its input value, and in fact, all output values of a function back to their input values. Many of the students expressed this core idea directly or indirectly in their responses.

The more challenging components seemed to be attending to the idea that it was a function – and not a subset of points, or only the points in a table. An inverse *function* must apply to all input-output pairs.

The other area that seemed challenging overall was for students to explain *why* a certain process would help them determine whether they had an inverse function. For example, we "switch the x and y, and solve for x, to get the inverse." This is true – and is a well-rehearsed rule. How does this process relate to the idea of inverse? To strengthen the argument, a student might explain that the process of "switching the x and y" means that I'm trying to create a function that takes all the outputs (the ys) from the original function, and when I put them into a new inverse function (as an x conventionally) it should produce the inputs from my original function (my original xs). It is no easy task to express this idea, but this connection between process and core idea of inverse is crucial for students to get their minds around for both the purposes of developing an understanding of a process, as well as for the purpose of producing a compelling argument that does not leave much for the reader to infer.